On Weyl channels being covariant with respect to the maximum commutative group of unitaries

Grigori G. Amosov*

Department of Higher Mathematics

Moscow Institute of Physics and Technology

Dolgoprudny 141700

RUSSIA

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Abstract

We investigate the Weyl channels being covariant with respect to the maximum commutative group of unitary operators. This class includes the quantum depolarizing channel and the "two-Pauli" channel as well. Then, we show that our estimation of the output entropy for a tensor product of the phase damping channel and the identity channel based upon the decreasing property of the relative entropy allows to prove the additivity conjecture for the minimal output entropy for the quantum depolarizing channel in any prime dimesnsion and for the "two Pauli" channel in the qubit case.

1   Introduction

Let H be a finite-dimensional Hilbert space. Denote $\sigma(H)$ the set of states (positive unit-trace operators) in H. The linear map Φ :

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*E-mail: gramos@mail.ru

$\sigma(H) \to \sigma(H)$ is said to be a quantum channel if Φ^* is completely positive ([8]). The quantum channel Φ is called bistochastic if $\Phi(\frac{1}{d}I_H) = \frac{1}{d}I_H$, where I_H is the identity operator in H.

The Holevo-Schumacher-Westmoreland bound $C_1(\Phi)$ of a quantum channel Φ is defined by the formula

$$C_1(\Phi) = \sup_{x_j \in \sigma(H), \pi} S(\sum_{j=1}^r \pi_j \Phi(x_j)) - \sum_{j=1}^r \pi_j S(\Phi(x_j)),$$

where $S(x) = -Trx\log x$ is the von Neumann entropy of x and the supremum is taken over all probability distributions $\pi = (\pi_j)_{j=1}^r$, $0 \le \pi_j \le 1$, $\sum_{j=1}^r \pi_j = 1$ for all $r \in \mathbb{N}$. *The additivity conjecture* states ([1]) that for any two channels Φ and Ψ

$$C_1(\Phi \otimes \Psi) = C_1(\Phi) + C_1(\Psi).$$

If the additivity conjecture holds, one can easily find the capacity $C(\Psi)$ of the channel Ψ by the formula $C(\Psi) = \lim_{n \to +\infty} \frac{C_1(\Psi^{\otimes n})}{n} = C_1(\Psi)$ (see [9]).

At the moment, the additivity conjecture is proved for a number of different cases ([9, 13, 14, 16, 5, 3]). The crucial role was played by C. King who proved [13, 14] the additivity conjecture for all bistochastic qubit channels as well as for the quanum depolarizing channel. In [2] it was shown that the additivity conjecture for the channels of the form $\Psi \circ \Phi$, where Φ is the quantum depolarizing channel and Ψ is the phase damping can be proved by means of the decreasing property of the relative entropy [15]. In the present paper we continue to study a possibility to use the decreasing property of the relative entropy to estimate the von Neumann entropy of the output of a quantum channel.

The additivity conjecture for the quantity C_1 is closely connected with the additivity conjecture for the entropy infimum which states that

$$\inf_{\rho \in \sigma(H \otimes K)} S((\Phi \otimes \Psi)(\rho)) = \inf_{\rho \in \sigma(H)} S(\Phi(\rho)) + \inf_{\rho \in \sigma(K)} S(\Psi(\rho))$$

for any two channels Φ and Ψ acting in Hilbert spaces H and K, respectively. For the case of the Weyl channel we consider in the present paper, if the additivity conjecture for the entropy infimum

holds it implies that the quantity C_1 is additive also ([10]). As it is shown in [17] the additivity conjectures for C_1 and the entropy infimum are globally equivalent.

Fix the basis e_j, $0 \le j \le d-1$, in a Hilbert space H, $dim H = d$, and define a unitary operator U by the formula

$$U = \sum_{j=0}^{d-1} e^{\frac{2\pi i}{d} j} |e_j >< e_j|.$$

Given a probability distribution $0 \le \lambda_j \le 1$, $\sum_{j=0}^{d-1} \lambda_j = 1$, we shall call the bistochastic quantum channel defined as

$$\Phi(x) = \sum_{j=0}^{d-1} \lambda_j U^j x U^{*j}, \ x \in \sigma(H), \tag{1}$$

by *a phase damping*. The map

$$E(x) = \frac{1}{d} \sum_{j=0}^{d-1} U^j x U^{*j} \equiv \sum_{j=0}^{d-1} |e_j >< e_j| x |e_j >< e_j|,$$

$x \in \sigma(H)$, is a conditional expectation to the algebra of the elements being fixed with respect to the action of the group $\{U^j, \ 0 \le j \le d-1\}$ and, hence, of the channel Φ.

Given two states ρ and $\overline{\rho}$, let $S(\rho, \overline{\rho}) = Tr(\rho \log \rho) - Tr(\rho \log \overline{\rho})$ be the Umegaki relative entropy [18]. Due to [15] the following property holds,

$$S(\Phi(\rho), \Phi(\overline{\rho})) \le S(\rho, \overline{\rho})$$

for any (not necessarily bistochastic) quantum channel Φ. Using this property it is possible to estimate the entropy of the output of a quantum channel ([2]).

Let K be an arbitary finite-dimensional Hilbert space. Using the decreasing property of the relative entropy we shall prove the following statement.

Theorem 1.*Let $x \in \sigma(H \otimes K)$ be such that*

$$E(Tr_K(x)) = \frac{1}{d} I_H.$$

Then, for the phase damping (1) we get

$$S((\Phi \otimes Id)(x)) \ge - \sum_{j=0}^{d-1} \lambda_j \log \lambda_j + \frac{1}{d} \sum_{j=0}^{d-1} S(x_j), \tag{2}$$

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where $x_j = dTr_H((|e_j><e_j| \otimes I_K)x) \in \sigma(K),\ 0 \leq j \leq d-1$.

Analyzing the Weyl channels being covariant with respect to the maximum commutative group of unitary operators we shall prove the following two theorems based upon the estimation (2).

Theorem 2. *Let $\Phi(\rho) = (1-p)\rho + \frac{p}{d}I_H,\ \rho \in \sigma(H),\ 0 \leq p \leq \frac{d^2}{d^2-1}$, be the quantum depolarizing channel in the Hilbert space H of the prime dimension d. Then, there exists d orthonormal bases $\{f_j^s,\ 0 \leq s, j \leq d-1\}$ in H such that*

$$S((\Phi \otimes Id)(x)) \geq -(1 - \frac{d-1}{d}p)\log(1 - \frac{d-1}{d}p) - \frac{d-1}{d}p\log\frac{p}{d} + \quad (3)$$

$$\frac{1}{d^2}\sum_{j=0}^{d-1}\sum_{s=0}^{d-1} S(x_j^s),$$

where $x \in \sigma(H \otimes K),\ x_j^s = dTr_H((|e_j^s><e_j^s| \otimes I_K)x) \in \sigma(K),\ 0 \leq j, s \leq d-1$.

Denote σ_x, σ_y and σ_z the Pauli operators.

Theorem 3. *Let $\Phi(\rho) = (1-2p)\rho + p\sigma_y\rho\sigma_y + p\sigma_z\rho\sigma_z$ be the "two Pauli" channel in the space H, $dimH = 2$. Suppose that $p \leq \frac{1}{3}$, then there exist three orthonormal bases $(e_1^s, e_2^s),\ 1 \leq s \leq 3$, in H such that*

$$S((\Phi \otimes Id)(x)) \geq -(1-p)\log(1-p) - p\log p + \frac{1}{6}\sum_{n=1}^{3}(S(x_j^1) + S(x_j^2)),$$

where $x \in \sigma(H \otimes K),\ x_j^k = 2Tr_H((|e_k^j><e_k^j| \otimes I_K)x) \in \sigma(K),\ 1 \leq j \leq 3,\ 1 \leq k \leq 2$.

Remark. *The additivity of the minimal output entropy for the quantum depolarizing channel and the "two-Pauli" channel follows from Theorems 2 and 3. Indeed, these theorems contain more information. It was shown in [13] that if the additivity conjecture holds for the qubit depolarizing channel and the "two-Pauli" channel in the case $p \leq \frac{1}{3}$, then it holds for all bistochastic qubit channels. Hence, in particular, Theorems 2 and 3 imply that the additivity conjecture holds for all bistochastic qubit channels.*

This paper is organized as follows. In Part 2 we prove Theorem 1. Part 3 is devoted to the description of the Weyl (the same as Weyl-covariant) channels. The orbits of the maximum commutative groups in the dimensions $d = 2$ and $d = 3$ are described in Part 3. In Part 4 we find the conditions which are sufficient for the Weyl channel be covariant with respect to the maximum commutative group. In Part 5 the proof of Theorems 2 and 3 is given.

2   Proof of Theorem 1

In this section we shall use the approach introduced in [2].

Fix x with the property formulated in Theorem 1 and define a quantum channel $\Xi_x : \sigma(H \otimes K) \to \sigma(H \otimes K)$ by the formula

$$\Xi_x(\rho) = \sum_{j=0}^{d-1} Tr((|e_j><e_j|\otimes I_K)\rho)(U^j\otimes I_K)x(U^{*j}\otimes I_K), \ \rho \in \sigma(H\otimes K).$$

Put $\rho = \sum_{j=0}^{d-1} \lambda_j |e_j><e_j| \otimes y$, $\overline{\rho} = \sum_{j=0}^{d-1} \frac{1}{d}|e_j><e_j| \otimes y = \frac{1}{d}I_H \otimes y$,
where $y \in \sigma(K)$ is an arbitrary fixed state. Then,

$$\Xi_x(\rho) = (\Phi \otimes Id)(x),$$

$$\Xi_x(\overline{\rho}) = \frac{1}{d}\sum_{j=0}^{d-1}(U^j \otimes I_K)x(U^{*j} \otimes I_K) \equiv \tilde{E}(x).$$

Notice that $\tilde{E} = (E \otimes Id)$ is the conditional expectation to algebra of the elements being fixed with respect to the action of the cyclic group $\{U^j \otimes I_K, \ 0 \leq j \leq d-1\}$.

The decreasing property of the relative entropy [15] gives us the estimation:

$$S(\Xi_x(\rho), \Xi_x(\overline{\rho})) \leq S(\rho, \overline{\rho}) = \sum_{j=0}^{d-1}\lambda_j \log \lambda_j + \log(d). \qquad (4)$$

On the other hand,

$$S(\Xi_x(\rho), \Xi_x(\overline{\rho})) = Tr((\Phi \otimes Id)(x)\log(\Phi \otimes Id)(x)) -$$

$$Tr((\Phi \otimes Id)(x)\log \tilde{E}(x)) = -S((\Phi \otimes Id)(x)) -$$

$$Tr(\tilde{E} \circ (\Phi \otimes Id)(x)\log \tilde{E}(x)) =$$

$$- S((\Phi \otimes Id)(x)) + S(\tilde{E}(x)). \qquad (5)$$

Here we used the equality $\tilde{E} \circ (\Phi \otimes Id) = \tilde{E}$ which holds because $\tilde{E}$ is the conditional expectation to the algebra of elements being fixed with respect to the action of $\Phi \otimes Id$. It follows from the condition of Theorem 1 that

$$\tilde{E}(x) = \frac{1}{d}\sum_{j=0}^{d-1}|e_j><e_j| \otimes x_j, \ x_j \in \sigma(K).$$

Thus,

$$S(\tilde{E}(x)) = log(d) + \frac{1}{d}\sum_{j=0}^{d-1} S(x_j), \tag{6}$$

$x_j = dTr_H((|e_j><e_j| \otimes I_K)x), \ 0 \le j \le d-1$. Combining (4), (5) and (6) we get the result we need.

3 The Weyl channels

Recently the representation of quantum channels by means of the discrete Weyl group were discussed in different contexts ([2, 4, 7, 12]). Fix the orthonormal basis $|k>, \ k = 0,1,\ldots,d-1$ of the Hilbert space $H, \ dimH = d$, and define the unitary operators $U_{m,n}$ by the formula ([6])

$$U_{m,n} = \sum_{k=0}^{d-1} e^{\frac{2\pi i}{d}kn}|k+m \ mod \ d><k|,$$

$0 \le m,n \le d-1$. The operators $U_{m,n}$ are satisfied the Weyl commutation relations,

$$U_{m,n}U_{m',n'} = e^{2\pi i(m'n-mn')/d}U_{m',n'}U_{m,n}, \tag{7}$$

$0 \le m,n,m',n' \le d-1$. We shall call $U_{m,n}$ by a Weyl operator. Notice that

$$U_{m,0}|k> = |k+m \ mod \ d>, \ U_{0,n}|k> = e^{\frac{2\pi i}{d}kn \ mod \ d}|k>, \tag{8}$$

$$U_{m,n} = U_mU_n, \ 0 \le m,n \le d-1.$$

We shall consider bistochastic quantum communication channels of the following form

$$\Phi(x) = \sum_{m,n=0}^{d-1} \pi_{m,n}U_{m,n}xU_{m,n}^*, \tag{9}$$

$x \in \sigma(H)$, where $0 \le \pi_{m,n} \le 1, \ \sum_{m,n=0}^{d-1} \pi_{m,n} = 1$, is an arbitrary probability distribution. We shall call (9) by a Weyl channel.

The bases $(e_j)_{j=0}^{d-1}$ and $(f_j)_{j=0}^{d-1}$ of the space H are said to be mutually unbiased bases ([11]) if $| < e_j|f_k > | = \frac{1}{\sqrt{d}}, \ 0 \le j,k \le d-1$. For

example, the eigenvectors $f_j = |j>$ and $e_j = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} e^{\frac{2\pi i}{d}jk}|k>$ of the Weyl operators $U_{0,n}$ and $U_{m,0}$, respectively, form mutually unbiased bases. If the dimension d of the Hilbert space H is prime, there exist $d+1$ mutually unbiased bases $\{e_j^k,\ 0 \leq j \leq d-1,\ 0 \leq k \leq d\}$ which can defined as the eigenvectors of the Weyl operators $U_{sk,k}$ such that

$$|e_j^s >< e_j^s| = \frac{1}{d} \sum_{k=0}^{d-1} e^{\frac{2\pi i}{d}jk} U_{sk,k},\ 0 \leq s \leq d-1, \qquad (10)$$

$$|e_j^d >< e_j^d| = \frac{1}{d} \sum_{k=0}^{d-1} e^{\frac{2\pi i}{d}jk} U_{k,0} \equiv |j >< j|.$$

Because the elements of the mutually unbiased bases (e_j^s) are eigenvectors for the Weyl operators $U_{sk,k}$ and $U_{k,0}$, we get

$$U_{sk,k} = \sum_{j=0}^{d-1} e^{\frac{2\pi i}{d}jk} |e_j^s >< e_j^s|,\ 0 \leq s \leq d-1,$$

$$U_{k,0} = \sum_{j=0}^{d-1} e^{\frac{2\pi i}{d}jk} |e_j^d >< e_j^d|.$$

Fix a basis $(e_j)_{j=0}^{d-1}$ in the space H. We shall call by the maximimum commutative group of unitaries $\mathcal{U}_d((e_j)_{j=0}^{d-1})$ a d-parameter group of unitary operators

$$U = \sum_{j=0}^{d-1} e^{i\phi_j} |e_j >< e_j|, \qquad (11)$$

where $\phi_j,\ 0 \leq j \leq d-1$, are arbitrary real numbers. Denote by $\mathcal{U}_d^s = \mathcal{U}_d((e_j^s)_{j=0}^{d-1}),\ 0 \leq s \leq d$, the maximum commutative groups generated by the mutually unbiased bases (10).

The bistochastic quantum channel

$$E_s(x) = \sum_{j=0}^{d-1} |e_j^s >< e_j^s| x | e_j^s >< e_j^s|,\ x \in \sigma(H),$$

is a conditional expectation to the algebra of elements being fixed with respect to the action of the maximum commutative group $\mathcal{U}_d^s$ as well

as of the phase damping channel

$$\Phi_s(x) = \sum_{j=0}^{d-1} \lambda_j U_{sj,j} x U_{sj,j}^*$$

for $0 \le s \le d-1$ and

$$\Phi_d(x) = \sum_{j=0}^{d-1} \lambda_j U_{j,0} x U_{j,0}$$

for $s = d$.

4 Orbits of the maximum commutative group of unitaries.

Denote by $\mathcal{G}_d^s$ the set consisting of the unit vectors g for which there exists the unitary operator $U_s \in \mathcal{U}_d^s$, $0 \le s \le d-1$, such that

$$E_d(U_s|g><g|U_s^*) = \frac{1}{d} I_H,$$

Let $\mathcal{A}_d^s$ be the convex set of states generated by the one-dimensional projections $|g><g|$, $g \in \mathcal{G}_d$.

Lemma 1. *Given complex numbers a, b, $|a|^2 + |b|^2 = 1$ there exist the real numbers $\phi, \psi, \ \alpha \in [0, 2\pi]$ such that*

$$e^{i\phi}a + e^{i\psi}b = 1,$$

$$e^{i\phi}a - e^{i\psi}b = e^{i\alpha}.$$

Proof.

Pick up $\tilde{\phi}, \tilde{\psi}$ such that $\cos \tilde{\phi} = \sin \tilde{\psi} = |a|$, $-\sin \tilde{\phi} = \cos \tilde{\psi} = |b|$. Then, $\phi = \tilde{\psi} - arg(a)$, $\psi = \tilde{\psi} - arg(b)$, $\cos \alpha = \cos(2\tilde{\phi})$, $\sin \alpha = \sin(2\tilde{\phi})$ give us the solution. $\square$

Proposition 1. *Suppose that $d = 2$. Then, $\mathcal{G}_2^0 = \mathcal{G}_2^1 = H$ and, hence, $\mathcal{A}_2^0 = \mathcal{A}_2^1 = \sigma(H)$.*

Proof.

We shall prove that $\mathcal{G}_2^0 = H$. The identity $\mathcal{G}_2^1 = H$ can be considered analogously.

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Take a vector $g \in H$. Suppose that the coordinates of g in the basis e_0^0, e_1^0 are $< e_0^0|g >= a$, $< e_1^0|g >= b$. Let the numbers $\phi, \psi, \alpha \in [0, 2\pi]$ are defined by Lemma 1. Determine the unitary operator $U \in \mathcal{U}_2^0$ by the formula

$$U = e^{i\phi}|e_0^0 >< e_0^0| + e^{i\psi}|e_1^0 >< e_1^0|,$$

then, it follows from Lemma 1, that

$$< e_0^2|Ug >= \frac{1}{\sqrt{2}}, \quad < e_1^2|Ug >= \frac{e^{i\alpha}}{\sqrt{2}}.$$

$\square$

Proposition 2. $\mathcal{G}_3^s \neq H$, $0 \leq s \leq 2$.

Proof.

We shall prove that $\mathcal{G}_3^0 \neq H$. The other cases can be considered analogously.

Take a unit vector $g \in H$ whose coordinates $< e_j^0|g >= \alpha_j$, $0 \leq j \leq 2$,. If $\mathcal{G}_3^0 = H$, then there exist the real numbers ϕ_j, β_j, $0 \leq j \leq 2$, satisfying the relations

$$\sum_{k=0}^{2} e^{i\phi_k} e^{i\frac{2\pi}{3}jk} = e^{i\beta_j}, \ 0 \leq j \leq 2. \tag{12}$$

Suppose that $\alpha_1, \alpha_2 > 0$ and $\alpha_3 = 0$, then the condition (12) implies that

$$\cos(\phi_1 - \phi_2) = 0,$$

$$\cos(\phi_1 - \phi_2 - \frac{2\pi}{3}) = 0$$

and

$$\cos(\phi_1 - \phi_2 - \frac{4\pi}{3}) = 0.$$

But the system of these three equation has no solution. $\square$

Lemma 2. *Given complex numbers* α_j, $\sum\limits_{j=0}^{2} |\alpha_j|^2 = 1$ *satisfying the relations*

$$|\alpha_1\alpha_2| + |\alpha_1\alpha_3| > |\alpha_2\alpha_3|,$$

$$|\alpha_1\alpha_2| + |\alpha_2\alpha_3| > |\alpha_1\alpha_3|,$$

$$|\alpha_1\alpha_3| + |\alpha_2\alpha_3| > |\alpha_1\alpha_2|.$$

there exist the real numbers $\phi_j, \beta_j, \ 0 \le j \le 2$, such that

$$\sum_{j=0}^{2} e^{i(\phi_j + \frac{2\pi jk}{3})} \alpha_j = e^{i\beta_k}, \ 0 \le k \le 2.$$

Proof.

Because $\alpha_j = |\alpha_j| e^{iarg(\alpha_j)}$, it suffices to consider only the case of non-negative real numbers $\alpha_j \ge 0, \ 0 \le j \le 2$. Denote by $\gamma_j, \ 0 \le j \le 2$, the angles of the triangle with the sides equal to the values $\alpha_1 \alpha_2, \ \alpha_1 \alpha_3$ and $\alpha_2 \alpha_3$. Then,

$$\frac{\sin \gamma_1}{\alpha_1 \alpha_2} = \frac{\sin \gamma_2}{\alpha_1 \alpha_3} = \frac{\sin \gamma_3}{\alpha_2 \alpha_3}, \tag{13}$$

$$\alpha_1 \alpha_2 = \alpha_1 \alpha_3 \cos(\gamma_3) + \alpha_2 \alpha_3 \cos(\gamma_2),$$
$$\alpha_1 \alpha_3 = \alpha_1 \alpha_2 \cos(\gamma_3) + \alpha_2 \alpha_3 \cos(\gamma_1), \tag{14}$$
$$\alpha_2 \alpha_3 = \alpha_1 \alpha_2 \cos(\gamma_2) + \alpha_1 \alpha_3 \cos(\gamma_1).$$

Put

$$\phi_1 = 0, \ \phi_2 = \frac{2\pi}{3} + \frac{1}{3}\gamma_2 - \frac{1}{3}\gamma_3,$$

$$\phi_3 = \frac{\pi}{3} - \frac{1}{3}\gamma_2 - \frac{2}{3}\gamma_3.$$

Thus, we get

$$\alpha_1 \alpha_2 \cos(\phi_1 - \phi_2) + \alpha_1 \alpha_3 \cos(\phi_1 - \phi_3) + \alpha_2 \alpha_3 \cos(\phi_2 - \phi_3) =$$

$$\alpha_1 \alpha_2 \cos(\phi_2) - \alpha_1 \alpha_3 \cos(\phi_2 + \gamma_3) - \alpha_2 \alpha_3 \cos(\phi_2 - \gamma_2) =$$

$$(\alpha_1 \alpha_2 - \alpha_1 \alpha_3 \cos(\gamma_3) - \alpha_2 \alpha_3 \cos(\gamma_2)) \cos(\phi_2) + \tag{15}$$

$$(\alpha_1 \alpha_3 \sin(\gamma_3) - \alpha_2 \alpha_3 \sin(\gamma_2)) \sin(\phi_2) = 0$$

due to (13) and (14). Analogously,

$$\alpha_1 \alpha_2 \sin(\phi_1 - \phi_2) - \alpha_1 \alpha_3 \sin(\phi_1 - \phi_3) + \alpha_2 \alpha_3 \sin(\phi_2 - \phi_3) =$$

$$-\alpha_1 \alpha_2 \sin \phi_2 + \alpha_1 \alpha_3 \sin(\phi_2 + \gamma_3) + \alpha_2 \alpha_3 \sin(\phi_2 - \gamma_2) =$$

$$(-\alpha_1 \alpha_2 + \alpha_1 \alpha_3 \cos(\gamma_3) + \alpha_2 \alpha_3 \cos(\gamma_2)) \sin(\phi_2) + \tag{16}$$

$$(\alpha_1 \alpha_3 \sin(\gamma_3) - \alpha_2 \alpha_3 \sin(\gamma_2)) \cos(\phi_2) = 0.$$

Denote
$$I_k = \sum_{j=0}^{2} e^{i(\phi_j + \frac{2\pi jk}{3})}\alpha_j, \ 0 \le k \le 2.$$

Under the condition $\sum_{j=0}^{2} \alpha_j^2 = 1$ the formulas (15)-(16) imply that
$$Re(I_k)^2 + Im(I_k)^2 = 1, \ 0 \le k \le 2.$$

□

Proposition 3. *The set $\mathcal{G}_3^s$ includes the unit vectors h whose coordinates α_1, α_2 and α_3 in the basis (e_j^s) satisfy the relation*
$$|\alpha_1\alpha_2| + |\alpha_1\alpha_3| > |\alpha_2\alpha_3|,$$
$$|\alpha_1\alpha_2| + |\alpha_2\alpha_3| > |\alpha_1\alpha_3|, \tag{17}$$
$$|\alpha_1\alpha_3| + |\alpha_2\alpha_3| > |\alpha_1\alpha_2|.$$

Remark. *We doesn't consider the trivial case $\alpha_i = \delta_{ii_0}$ for some $i_0, 0 \le i_0 \le 2$.*

Proof.

As it was done above we shall consider only the case $s = 0$.

Take a vector g satisfying the relations (17). Suppose that the coordinates of g in the basis e_0^0, e_1^0, e_2^0 are $< e_k^0|g >= \alpha_k, \ 0 \le k \le 2$. Let the numbers $\phi_j, \ 0 \le j \le 2$, are defined by Lemma 2. Determine the unitary operator $U \in \mathcal{U}_3^0$ by the formula
$$U = \sum_{j=0}^{2} e^{i\phi_j}|e_j^0 >< e_j^0|,$$

then, it follows from Lemma 2 that
$$| < e_j^3|Ug > | = \frac{1}{\sqrt{3}}, \ 0 \le j \le 2.$$

□

Denote by M_d and $M(0)$ the algebra of all $d \times d$ matrices and its subalgebra generated by the cyclic group $\{U_{m,0}, \ 0 \le n \le d-1\}$, respectively. Let $\mathcal{U}_d$ be the maximum commutative group constructed by means of the basis $(e_j)_{j=0}^{d-1}$.

Proposition 4. *The inclusion $W \in M(0)$ holds for all $W \in \mathcal{U}_d$.*

Proof.

It suffices to notice that $U_{m,0} \in \mathcal{U}_d, \ 0 \le m \le d-1$.

□

5 The covariant Weyl channels.

The quantum channel Φ is said to be covariant with respect to the group $\mathcal{U}$ being a subgroup of the group of all unitary operators in H if

$$\Phi(UxU^*) = U\Phi(x)U^*, \ x \in \sigma(H), \ U \in \mathcal{U}.$$

The Weyl operators $U_{m,n}$ satisfying the relation (7) form the basis in the algebra of all $d \times d$ matrices. Moreover, the action of a Weyl channel (9) on the Weyl operator $U_{s,t}$ is given by the formula

$$\Phi(U_{s,t}) = \lambda_{st} U_{s,t},$$

where

$$\lambda_{st} = \sum_{m,n=0}^{d-1} \pi_{m,n} e^{2\pi i(sn-tm)/d}.$$

Proposition 5. *Suppose that $\lambda_{st} = \mu_t = const$, $0 \leq s \leq d-1$, $1 \leq t \leq d-1$. Then the Weyl channel is covariant with respect to the maximum commutative group $\mathcal{U}_d^d$.*

Remark. *If $d = 2$, a role of the maximum commutative group of unitaries can be played by all multiples of the group $SO(2)$ consisting of all rotations in H implemented by the matrices of the form*

$$e^{i\psi} \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}, \ \phi, \psi \in [0, 2\pi].$$

It is straightforward to check that the bistochastic qubit channel determined by the triple $[\lambda_1, \lambda_2, \lambda_3]$ is covariant with respect to the group of rotations iff $\lambda_1 = \lambda_3$. Thus, this class includes the quantum depolarizing channel $[\lambda, \lambda, \lambda]$ and the "two-Pauli" channel $[\lambda, 1 - 2\lambda, \lambda]$ as well.

Proof.

Denote $M(n)$ the algebra of matrices generated by the Weyl operators $U_{m,n}$, $0 \leq m \leq d-1$. Due to Proposition 4 a unitary operator $W \in \mathcal{U}_d^d$ can be represented as

$$W = \sum_{m=0}^{d-1} c_m U_{m,0}.$$

Hence $WU_{m,0}W^* = U_{m,0}$, $0 \leq m \leq d-1$, and $WM(n)W^* \subset M(n)$ for all $W \in \mathcal{U}_d^d$, $1 \leq n \leq d-1$. It follows that if the condition of

Proposition 5 holds, we get $WU_{s,t}W^* \in M(t)$ for $1 \leq t \leq d - 1$. Hence,

$$\Phi(WU_{s,t}W^*) = \mu_t WU_{st}W^* = W\Phi(U_{s,t})W^*,$$

$0 \leq s, t \leq d - 1$. $\square$

Proposition 6. *Suppose that $\pi_{mn} = p_m$, $0 \leq m \leq d - 1$, $1 \leq n \leq d - 1$. Then the Weyl channel (9) is covariant with respect to the maximum commutative group $\mathcal{U}_d^d$.*

Remark. *If $\pi_{00} = (1 - \frac{d^2-1}{d^2}p)$ and $\pi_{m0} = p_n = \frac{p}{d^2}$, $1 \leq n, m \leq d - 1$, the Weyl channel (9) is the quantum depolarizing channel,*

$$\Phi(x) = (1 - p)x + \frac{p}{d}Tr(x), \ 0 \leq p \leq \frac{d^2}{d^2 - 1}.$$

Suppose that $d = 2$, then the "two Pauli" channel

$$\Phi(\rho) = (1 - 2p)\rho + p\sigma_y\rho\sigma_y + p\sigma_z\rho\sigma_z, \ \rho \in \sigma(H), \quad (18)$$

is satisfied the conditions of Proposition 6 if we identify the Weyl operators with the Pauli operators such that $U_{0,0} \equiv I_H$, $U_{1,0} \equiv \sigma_x$, $U_{0,1} \equiv \sigma_y$, $U_{1,1} \equiv i\sigma_z$.

Proof.

In the case,

$$\lambda_{st} = \sum_{m=0}^{d-1} \pi_{m0}e^{-2\pi itm/d} + \sum_{n=1}^{d-1}\sum_{m=0}^{d-1} p_m e^{2\pi i(sn-tm)/d} =$$

$$\sum_{m=0}^{d-1}(\pi_{m0} - p_m)e^{-2\pi itm/d} \equiv \mu_t,$$

$0 \leq s \leq d - 1$, $1 \leq t \leq d - 1$. So, the result follows from Proposition 5.

$\square$

Fix the positive numbers $0 \leq p_n \leq 1$, $0 \leq r_m \leq 1$, $1 \leq n \leq d - 1$, $0 \leq m \leq d - 1$, such that $d\sum_{n=1}^{d-1} p_n + \sum_{m=0}^{d-1} r_m = 1$ and consider the Weyl channel

$$\Phi(x) = \sum_{m=0}^{d-1} r_m U_{m,0}xU_{m,0}^* + \sum_{m=0}^{d-1}\sum_{n=1}^{d-1} p_n U_{m,n}xU_{m,n}^*, \quad (19)$$

$x \in \sigma(H)$.

Proposition 7. *Suppose that d is a prime number, then the Weyl channel (19) can be represented in the form*

$$\Phi(x) = \sum_{k=0}^{d-1} \sum_{m=0}^{d-1} c_m U_{m,0} \Phi_k(x) U_{m,0}^*, \tag{20}$$

where

$$\Phi_k(x) = \sum_{n=0}^{d-1} \lambda_n U_{nk \bmod d, n} x U_{nk \bmod d, n}^*, \ x \in \sigma(H),$$

are the phase damping channels and

$$\lambda_0 = 1 - d \sum_{n=1}^{d-1} p_n, \ \lambda_n = dp_n, 1 \le n \le d-1,$$

$$c_m = \frac{r_m}{d(1 - d\sum\limits_{n=1}^{d-1} p_n)}, \ 0 \le m \le d-1.$$

Remark. *Suppose that $d = 2$, then the qubit Weyl channel (19) has the following form,*

$$\Phi(\rho) = r_0 \rho + r_1 \sigma_x \rho \sigma_x + p_1 \sigma_y \rho \sigma_y + p_1 \sigma_z \rho \sigma_z, \ \rho \in \sigma(H),$$

where we have identified the Weyl operators with the Pauli matrices such that $U_{0,0} \equiv I_H$, $U_{1,0} \equiv \sigma_x$, $U_{0,1} \equiv \sigma_y$, $U_{1,1} \equiv i\sigma_z$. In the case, in the representation (20) we get

$$\Phi(\rho) = \frac{r_0}{2(1 - 2p_1)} (\Phi_0(\rho) + \Phi_1(\rho)) + \frac{r_1}{2(1 - 2p_1)} \sigma_x (\Phi_0(\rho) + \Phi_1(\rho)) \sigma_x,$$

$$\Phi_0(\rho) = (1 - 2p_1)\rho + 2p_1 \sigma_y \rho \sigma_y,$$

$$\Phi_1(\rho) = (1 - 2p_1)\rho + 2p_1 \sigma_z \rho \sigma_z,$$

$\rho \in \sigma(H)$.

Proof.

Let us compare the equations (19) and (20). In (20) it is included d terms $c_0 \lambda_0 x$, d terms $c_m \lambda_0 U_{m,0} x U_{m,0}^*$ as well as d terms

$$c_{m-nk \bmod d} \lambda_n U_{m,n} x U_{m,n}^*, \ n \ne 0,$$

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(corresponding to each the channel Φ_k). Hence,

$$dc_m\lambda_0 = r_m, \ 0 \le m \le d-1,$$

$$\sum_{k=0}^{d-1} c_{m-nk \ mod \ d}\lambda_n = \sum_{k=0}^{d-1} c_k\lambda_n = dp_n, \ 1 \le n \le d-1.$$

Then, let us claim $\sum_{n=0}^{d-1} \lambda_n = \sum_{n=0}^{d-1} c_n = 1$. It follows that

$$dc_m\lambda_0 = r_m, \ 0 \le m \le d-1,$$

$$\lambda_n = dp_n, \ 1 \le n \le d-1.$$

Hence,

$$\lambda_0 = 1 - d\sum_{n=1}^{d-1} p_n,$$

$$c_m = \frac{r_m}{d(1 - d\sum_{n=1}^{d-1} p_n)}, \ 0 \le m \le d-1.$$

$\square$

6 The estimation of the output entropy.

Using Theorem 1 and Propositions 7 we can prove Theorem 2.

Proof of Theorem 2.

For $x \in \sigma(H \otimes K)$ choose the unitary operator W in H such that $Tr_K(y) = WTr_K(x)W^* \in \mathcal{A}_{fix}$, where $y = (W \otimes I_K)x(W^* \otimes I_K)$. Here we denoted $\mathcal{A}_{fix}$ the algebra of elements being fixed with respect to the action of the group $\mathcal{U}_d^d$. The algebra $\mathcal{A}_{fix}$ is generated by the projections $|e_j^d >< e_j^d|$. Hence,

$$E_s(Tr_K(y)) = \frac{1}{d}I_H, \ 0 \le s \le d-1,$$

because the bases (e_j^s) are mutually unbiased. Using the covariance of Φ with respect to the group of all unitary operators in H we get

$$S((\Phi \otimes Id)(x)) = S((\Phi \otimes Id)(y)).$$

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Due to Proposition 7 the tensor product of the quantum depolarizing channel and the identity channel can be represented as follows

$$(\Phi \otimes Id)(y) = \sum_{k=0}^{d-1} \sum_{m=0}^{d-1} c_m (U_{m,0} \otimes I_K)(\Phi_k \otimes Id)(y)(U_{m,0}^* \otimes I_K), \quad (21)$$

where

$$(\Phi_k \otimes Id)(y) = \sum_{n=0}^{d-1} \lambda_n (U_{nk \ mod \ d,n} \otimes I_K) y (U_{nk \ mod \ d,n}^* \otimes I_K),$$

$$\lambda_0 = 1 - \frac{d-1}{d} p, \ \lambda_n = \frac{p}{d}, 1 \leq n \leq d-1,$$

$$c_0 = \frac{1 - \frac{d^2-1}{d^2} p}{d(1 - \frac{d-1}{d} p)},$$

$$c_m = \frac{p}{d^3(1 - \frac{d-1}{d} p)}, \ 1 \leq m \leq d-1.$$

Applying Theorem 1 to each term $(\Phi_k \otimes Id)(y)$ included in the sum (21) we obtain the estimation (3) with $x_j^s = dTr((|f_j^s \rangle \langle f_j^s| \otimes I_K)x)$, where $|f_j^s \rangle = W^*|e_j^s \rangle$, $0 \leq j, s \leq d-1$.
□

Now consider the qubit case $d = 2$. Let us involve the Pauli operators such that

$$U_{0,0} \equiv I_H, \ U_{1,0} \equiv \sigma_x, \ U_{0,1} \equiv \sigma_y, \ U_{1,1} \equiv i\sigma_z.$$

Then, the maximum commutative groups $\mathcal{U}_2^0 \equiv \mathcal{U}_y, \ \mathcal{U}_2^1 \equiv \mathcal{U}_z, \ \mathcal{U}_2^2 \equiv \mathcal{U}_x$ are generated by the spectral projections of the Pauli operators σ_y, σ_z and σ_x, respectively. In the following we shall also use the notation $e_j^0 = e_j^y, \ e_j^1 = e_j^z$ and $e_j^2 = e_j^x, \ j = 0, 1$, as well as $E_0 = E_y, \ E_1 = E_z, \ E_2 = E_x$ for the corresponding conditional expectations.

The "two-Pauli" channel

$$\Phi(\rho) = (1 - 2p)\rho + p\sigma_y \rho \sigma_y + p\sigma_z \rho \sigma_z, \ \rho \in \sigma(H), \ 0 < p < \frac{1}{2}. \quad (22)$$

is covariant with respect to $\mathcal{U}_x$ by means of Proposition 6.

Proposition 8. *The channel (22) can be represented as a convex combination of the form*

$$\Phi(\rho) = \frac{1 - 3p}{1 - p} \Phi_1(\rho) + \frac{2p}{1 - p} \sigma_z \Psi_1(\rho) \sigma_z,$$

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where

$$\Phi_1(\rho) = (1 - p)\rho + p\sigma_y\rho\sigma_y$$

is the phase damping and the channel

$$\Psi_1(\rho) = p\rho + \frac{1 - p}{2}\sigma_x\rho\sigma_x + \frac{1 - p}{2}\sigma_z\rho\sigma_z$$

is covariant with respect to $\mathcal{U}_y$.

Proof.

It is straightforward to check the validity of the formula. To show that the channel Ψ_1 is covariant with respect to $\mathcal{U}_y$ let us redefine the correspondence between the Pauli operators and the Weyl operators such that

$$\sigma_y \equiv U_{1,0}, \ \ \sigma_z \equiv U_{0,1}, \ \ \sigma_x \equiv -iU_{1,1}.$$

Then, the result follows from Proposition 6.

□

In the proof of Theorem 3 we shall need the "two Pauli" channel of the form

$$\Psi_1(\rho) = p\rho + \frac{1 - p}{2}\sigma_x\rho\sigma_x + \frac{1 - p}{2}\sigma_z\rho\sigma_z, \ \rho \in \sigma(H). \qquad (23)$$

It is straightforward to check that the following estimation holds.

Proposition 9. *Suppose that $p \leq \frac{1}{3}$. Then, the channel (23) is a convex combination of the form*

$$\Psi_0(\rho) = \frac{p}{1 - p}\Phi_0(\rho) + \frac{1 - 3p}{2(1 - 2p)}\sigma_x\Phi_1(\rho)\sigma_x +$$

$$(1 - \frac{p}{1 - p} - \frac{1 - 3p}{2(1 - 2p)})\sigma_z\Phi_1(\rho)\sigma_z,$$

where

$$\Phi_0(\rho) = (1 - p)\rho + p\sigma_x\rho\sigma_x,$$

$$\Phi_1(\rho) = (1 - p)\rho + p\sigma_y\rho\sigma_y,$$

$\rho \in \sigma(H)$, are two phase damping channels.

Proof of Theorem 3.

Fix the state $\rho \in \sigma(H \otimes K)$. Due to Proposition 1 one can find the unitary operator $W \in \mathcal{U}_x$ such that $Tr_K(\tilde{\rho}) = WTr_K(\rho)W^*$ satisfies the identity

$$E_y(Tr_K(\tilde{\rho})) = \frac{1}{2}I_H, \qquad (24)$$

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where $\tilde{\rho} = (W \otimes I_K)\rho(W^* \otimes I_K)$. Using the covariance of the channel Φ with respect to $\mathcal{U}_x$ we obtain

$$S((\Phi \otimes Id)(\rho)) = S((\Phi \otimes Id)(\tilde{\rho})).$$

It follows from Proposition 8 that

$$(\Phi \otimes Id)(\tilde{\rho}) = \frac{1-3p}{1-p}(\Phi_1 \otimes Id)(\tilde{\rho}) + \frac{2p}{1-p}(\sigma_z \otimes I_K)(\Psi_1 \otimes Id)(\tilde{\rho})(\sigma_z \otimes I_K),$$

where

$$(\Phi_1 \otimes Id)(\tilde{\rho}) = (1-p)\tilde{\rho} + p(\sigma_y \otimes I_K)\tilde{\rho}(\sigma_y \otimes I_K),$$

and

$$(\Psi_1 \otimes Id)(\tilde{\rho}) = p\tilde{\rho} + \frac{1-p}{2}(\sigma_x \otimes I_K)\tilde{\rho}(\sigma_x \otimes I_K) + \frac{1-p}{2}(\sigma_z \otimes I_K)\tilde{\rho}(\sigma_z \otimes I_K).$$

Hence, we get

$$S((\Phi \otimes Id)(\tilde{\rho})) \geq \min\{S(\Phi_1 \otimes Id)(\tilde{\rho}), \ S((\Psi_1 \otimes Id)(\tilde{\rho}))\}. \tag{25}$$

Taking into account (24) one can apply Theorem 1 to $(\Phi_1 \otimes Id)(\tilde{\rho})$. Then,

$$S((\Phi_1 \otimes Id)(\tilde{\rho})) \geq -(1-p)\log(1-p) - p\log p + \frac{1}{2}(S(2Tr_H((|e_1^1 >< e_1^1| \otimes I_K)\rho)) +$$

$$S(2Tr_H((|e_2^1 >< e_2^1| \otimes I_K)\rho)), \tag{26}$$

where $|e_1^1> = W^*|e_0^y>$, $|e_2^1> = W^*|e_1^y>$.

It follows from Proposition 1 that there exists the unitary operator $\tilde{W} \in \mathcal{U}_y$ such that $Tr_K(\tilde{\tilde{\rho}}) = \tilde{W}Tr_K(\tilde{\rho})\tilde{W}^*$ satisfies the identity

$$E_x(Tr_K(\tilde{\tilde{\rho}})) = \frac{1}{2}I_H, \tag{27}$$

where $\tilde{\tilde{\rho}} = (\tilde{W} \otimes I_K)\tilde{\rho}(\tilde{W}^* \otimes I_K)$. Moreover,

$$E_y(Tr_K(\tilde{\tilde{\rho}})) = E_y(\tilde{W}Tr_K(\tilde{\rho})\tilde{W}^*) = E_y(Tr_K(\tilde{\rho})) = \frac{1}{2}I_H, \tag{28}$$

because $\tilde{W}$ belongs to the algebra generated by the projections $|e_0^y >< e_0^y|$, $|e_1^y >< e_1^y|$, such that $E_y(\tilde{W}) = \tilde{W}$.

The covariance of Ψ_1 with respect to $\mathcal{U}_y$ gives us

$$S((\Psi_1 \otimes Id)(\tilde{\rho})) = S((\Psi_1 \otimes Id)(\tilde{\tilde{\rho}})).$$

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It follows from Proposition 9 that the channel $\Psi_1 \otimes Id$ can be represented in the form

$$(\Psi_1 \otimes Id)(\tilde{\rho}) = \frac{p}{1-p}(\Phi_0 \otimes Id)(\tilde{\rho}) + \frac{1-3p}{2(1-2p)}(\sigma_x \otimes I_K)(\Phi_1 \otimes Id)(\tilde{\rho})(\sigma_x \otimes I_K) + \tag{29}$$

$$(1 - \frac{p}{1-p} - \frac{1-3p}{2(1-2p)})(\sigma_z \otimes I_K)(\Phi_1 \otimes Id)(\tilde{\rho})(\sigma_z \otimes I_K),$$

where

$$(\Phi_0 \otimes Id)(\tilde{\rho}) = (1-p)\tilde{\rho} + p(\sigma_x \otimes I_K)\tilde{\rho}(\sigma_x \otimes I_K),$$

$$(\Phi_1 \otimes Id)(\tilde{\rho}) = (1-p)\tilde{\rho} + p(\sigma_y \otimes I_K)\tilde{\rho}(\sigma_y \otimes I_K),$$

Taking into account the identities (27) and (28) one can apply Theorem 1 to $(\Phi_0 \otimes Id)(\tilde{\rho})$ and $(\Phi_1 \otimes Id)(\tilde{\rho})$. Thus, we obtain

$$S((\Phi_0 \otimes Id)(\tilde{\rho})) \geq -(1-p)\log(1-p) - p\log p + \frac{1}{2}(S(2Tr_H((|e_1^2 >< e_1^2| \otimes I_K)\rho)) + \tag{30}$$

$$S(2Tr_H((|e_2^2 >< e_2^2| \otimes I_K)\rho))),$$

where $|e_1^2 >= W^*\tilde{W}^*|e_0^x >$, $|e_2^2 >= W^*\tilde{W}^*|e_1^x >$ and

$$S((\Phi_1 \otimes Id)(\tilde{\rho})) \geq -(1-p)\log(1-p) - p\log p + \frac{1}{2}(S(2Tr_H((|e_1^3 >< e_1^3| \otimes I_K)\rho)) + \tag{31}$$

$$S(2Tr_H((|e_2^3 >< e_2^3| \otimes I_K)\rho))),$$

where $|e_1^3 >= W^*\tilde{W}^*|e_0^y >$, $|e_2^3 >= W^*\tilde{W}^*|e_1^y >$. Combining (25), (26), (29), (30) and (31) we obtain the result.
□

Acknowledgments

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